INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11, 12

Tournament 43, Northern Fall 2021 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. The wizards A, B, C and D know that the integers $1, 2, \ldots, 12$ are written on 12 cards, one integer on each card, and that each wizard will get three cards and will see only their own cards. Having received the cards, the wizards made several statements in the following order.

   A: "One of my cards contains the number 8".

   B: "All my numbers are prime".

   C: "All my numbers are composite and they all have a common prime divisor".

   D: "Now I know all the cards of each wizard".

   What cards did A receive if everyone told the truth? (5 points)

2. A rook is initially placed on any square of a 10×10 chessboard. On each turn, it moves to an adjacent square (rooks can not move diagonally). After a number of turns, it has visited each square exactly once. Prove that for each main diagonal (the diagonal between the corners of the board) the following statement is true: in the rook's path there were two consecutive turns in which the rook first stepped away from the diagonal and then returned back to the diagonal. (7 points)

3. Grasshopper Gerald and his 2020 friends play leapfrog on a plane as follows. On each turn, Gerald jumps over a friend so that his original point and his resulting point are symmetric with respect to this friend. Gerald wants to perform a series of jumps such that he jumps over each friend exactly once. Let us say that a point is *achievable* if Gerald can finish the 2020th jump in it. What is the maximum number N such that for some initial placement of the grasshoppers there are exactly N achievable points? (7 points)

4. What is the minimum k for which among any three nonzero real numbers there are two numbers a and b such that either $|a-b| \leq k$ or $|\frac{1}{a} - \frac{1}{b}| \leq k$? (7 points)

5. Let $ABCD$ be a parallelogram and let P be a point inside it such that $\angle PDA = \angle PBA$. Let ω_1 be the excircle of the triangle PAB opposite to the vertex A. Let ω_2 be the incircle of the triangle PCD. Prove that one of the common tangents of ω_1 and ω_2 is parallel to AD. (9 points)

6. 20 buns with jam and 20 buns with treacle are arranged in a row in random order. Alice and Bob take turns taking one bun from either end of the row. Alice starts, and wants to finally obtain exactly 10 buns of both types; Bob tries to prevent this. Is it true that for any order of the buns, Alice can win no matter what Bob does? (10 points)

7. A square grid of size 2×2 is covered by two triangles. Is it necessarily true that:

(a) at least one of its four cells is fully covered by one of the triangles; (6 points)

(b) some square of size 1×1 can be placed into one of these triangles? (6 points)